

A NEGATIVE LAW OF ROBUST PERSISTENCE

**A Necessary Retention–Loss Bound for Dissipative
Systems under Antagonistic Forcing**

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Abstract

We establish a negative law of robust persistence for dissipative systems subject to antagonistic forcing.

Rather than providing a sufficient condition for stability or invariance, the result identifies a necessary bound that no control strategy can overcome.

For a minimal mechanical template admitting an energy storage function and distinct channels for actuation and disturbance, we prove that robust persistence inside a bounded energy region is impossible whenever the antagonistic channel possesses strictly greater boundary leverage than the combined effect of dissipation and actuation.

The result is formulated as a retention–loss inequality, derived from robust viability theory and expressed entirely in physically interpretable quantities.

A tight two–dimensional mechanical example demonstrates that the bound is not an artifact of analysis but a real operational limit, saturable up to an arbitrarily small margin.

1. Introduction

Persistence of dynamical systems under external forcing is often studied through sufficient conditions: stability, passivity, input–to–state stability, or barrier certificates.

These approaches answer the question of how to design a controller that succeeds.

Here we address a different and more fundamental question:

What cannot be done?

Specifically, we ask whether there exist intrinsic physical limits preventing any control strategy from maintaining persistence against worst–case antagonistic forcing, regardless of feedback design.

We answer this question affirmatively by establishing a negative law of robust persistence.

The result does not prescribe a controller.

It identifies a boundary that no controller can cross.

2. Minimal Mechanical Template

We consider the following mechanical system:

$$M \ddot{q} + D \dot{q} + \text{grad } U(q) = B_u u(t) + B_w w(t)$$

where:

- q is in $\mathbb{R}^n$,
- M is a constant symmetric positive definite mass matrix,
- D is a constant symmetric positive semidefinite dissipation matrix,
- $U(q)$ is a C^1 radially coercive potential,
- $u(t)$ is the control input,
- $w(t)$ is an antagonistic disturbance.

The total mechanical energy is defined as:

$$E(q, v) = 1/2 v^T M v + U(q)$$

with $v = \dot{q}$.

We define the admissible operating region:

$$K = \{ (q, v) : E(q, v) \leq E_{\max} \}$$

3. Energy Balance

Along any absolutely continuous trajectory, the time derivative of the energy satisfies:

$$dE/dt = - v^T D v + v^T B_u u(t) + v^T B_w w(t)$$

This identity is exact and does not depend on any gyroscopic or Coriolis terms, which cancel in the energy balance.

4. Robust Persistence and Viability

Robust persistence is defined as the existence of a control law $u(t)$, measurable and bounded, such that:

For all admissible disturbances $w(t)$, the trajectory remains in K for all time.

By robust viability theory, this is possible only if, at every boundary point (q, v) satisfying $E(q, v) = E_{\max}$, there exists a control action counteracting all admissible disturbances so that the energy flux does not point outward.

Formally, the necessary boundary condition is:

$$\inf_{\text{over admissible } u} \sup_{\text{over admissible } w} \frac{dE}{dt} \leq 0$$

on the boundary of K .

5. Retention–Loss Boundary Condition

Assume pointwise bounds:

$$|u(t)| \leq u_{\max}$$

$$|w(t)| \leq w_{\max}$$

Define the boundary flux functional:

$$\Psi(v) = -v^T D v$$

$$- u_{\max} * |B_u^T v|$$

$$+ w_{\max} * |B_w^T v|$$

Robust persistence inside K requires:

$$\Psi(v) \leq 0$$

for all v such that $E(q, v) = E_{\max}$.

This condition is purely local at the boundary and independent of any particular control strategy.

6. Retention–Loss Indices

Define the global retention and loss indices:

$$R = \inf \text{ over boundary } (v^T D v + u_{\max} * |B_u^T v|)$$

$$L = \sup \text{ over boundary } (w_{\max} * |B_w^T v|)$$

Then the boundary condition is equivalent to the global inequality:

$$R \geq L$$

This inequality constitutes a necessary condition for robust persistence.

7. Negative Law of Robust Persistence

Theorem.

For the mechanical system defined above, robust persistence inside K is impossible if:

$$R < L$$

Equivalently, if there exists a boundary velocity v such that:

- $$v^T D v - u_{\max} * |B_u^T v| + w_{\max} * |B_w^T v| > 0$$

then for any admissible control law $u(t)$, there exists an admissible disturbance $w(t)$ that forces exit from K in finite time.

This result is a negative law: it specifies a physical impossibility, not a design method.

8. Tight Two-Dimensional Example

Consider a two-dimensional system with:

$$M = I$$

$$D = \text{diag}(d1, d2)$$

$$U(q) = 1/2 k |q|^2$$

$$B_u = (1, 0)^T$$

$$B_w = (\cos(\theta), \sin(\theta))^T$$

The energy boundary corresponds to velocities of the form:

$$v(\phi) = V (\cos(\phi), \sin(\phi))$$

$$\text{with } V = \sqrt{2 E_{\max}}.$$

Define:

$$\Psi(\phi) = - V^2 (d1 \cos^2(\phi) + d2 \sin^2(\phi))$$

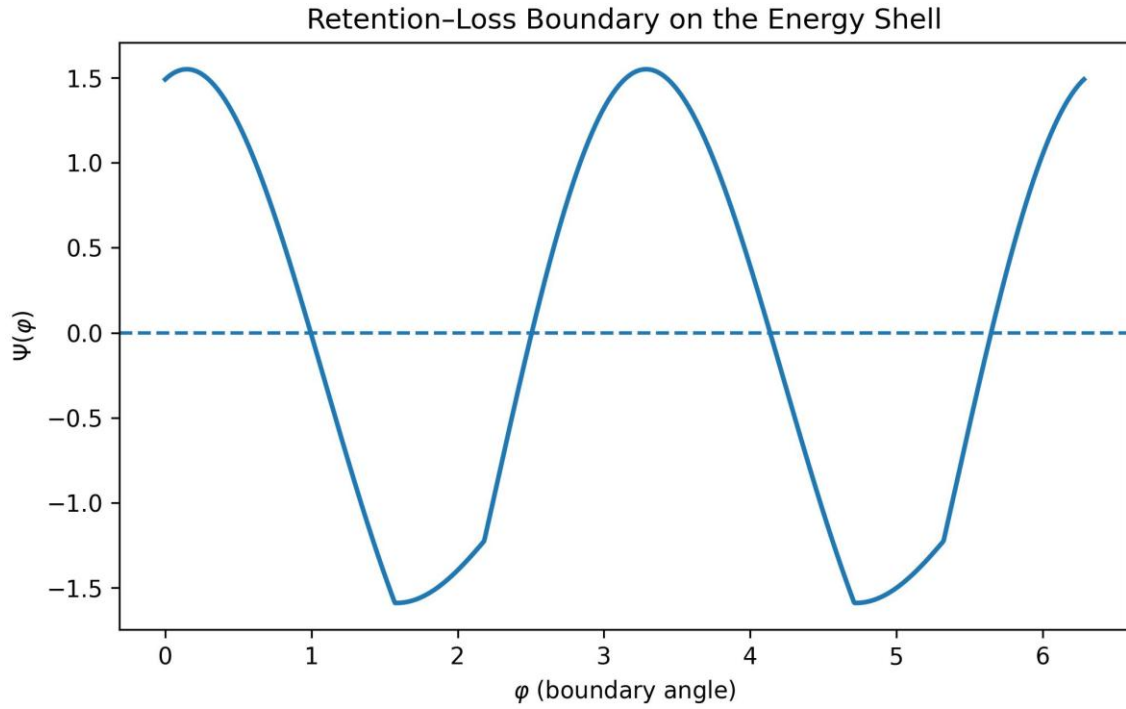
$$- u_{\max} * V * |\cos(\phi)|$$

$$+ w_{\max} * V * |\cos(\phi) \cos(\theta) + \sin(\phi) \sin(\theta)|$$

The function $\Psi(\phi)$ is continuous on a compact set.

By appropriate parameter choice, its maximum can be made arbitrarily close to zero from above.

This demonstrates that the bound $R \geq L$ is tight.



9. Finite-Time Exit

If there exists ϕ_{star} such that $\Psi(\phi_{\text{star}}) > 0$, then by continuity there exists a neighborhood where $\Psi \geq \epsilon > 0$.

Choosing the disturbance:

$$w(t) = w_{\text{max}} * \text{sign}(B_w^T v(t))$$

yields a strictly positive energy derivative over a set of times with positive measure, implying:

$$E(t) > E_{\text{max}}$$

in finite time.

10. Discussion and Positioning

This result is not a barrier certificate.

Barrier methods provide sufficient conditions for safety.

Here we derive a necessary condition for impossibility.

It is not passivity.

Passivity concerns stability under interconnection.

Here we identify an operational limit under antagonism.

The law is structural: any theory claiming robust persistence in the presence of antagonistic power injection must satisfy the retention–loss inequality at the boundary.

11. Conclusion

We have derived a negative law of robust persistence for dissipative systems with antagonistic forcing.

The result identifies a sharp physical limit beyond which no control strategy can succeed.

The law is necessary, local at the boundary, and tight.

Appendix A. Robust Viability and Finite-Time Exit

This appendix provides the formal justification of the negative law of robust persistence stated in the main text.

We consider the mechanical system:

$$M \ddot{q} + D \dot{q} + \text{grad } U(q) = B_u u(t) + B_w w(t)$$

with energy function:

$$E(q, v) = 1/2 v^T M v + U(q)$$

and admissible region:

$$K = \{ (q, v) : E(q, v) \leq E_{\max} \}.$$

A.1 Boundary Invariance and Robust Viability

Let (q, v) be a boundary point of K , i.e. $E(q, v) = E_{\max}$.

Robust persistence requires that for every admissible disturbance $w(t)$, there exists an admissible control $u(t)$ such that the trajectory does not leave K .

By robust viability theory for differential inclusions, a necessary condition for K to be invariant is that the vector field admits at least one inward-pointing direction at every boundary point.

Formally, the following condition must hold:

$$\inf_{\text{over admissible } u} \sup_{\text{over admissible } w} \frac{dE}{dt}(q, v; u, w) \leq 0$$

for all (q, v) such that $E(q, v) = E_{\max}$.

This condition is independent of any particular feedback law and depends only on the structure of the system and the admissible input sets.

A.2 Energy Flux Functional

Using the exact energy balance:

$$dE/dt = - \mathbf{v}^T \mathbf{D} \mathbf{v} + \mathbf{v}^T \mathbf{B}_u \mathbf{u} + \mathbf{v}^T \mathbf{B}_w \mathbf{w}$$

and assuming pointwise bounds:

$$|u(t)| \leq u_{\max}$$

$$|w(t)| \leq w_{\max}$$

the worst-case disturbance maximizes $\mathbf{v}^T \mathbf{B}_w \mathbf{w}$ by choosing $\mathbf{w}$ aligned with $\mathbf{B}_w^T \mathbf{v}$, while the best control minimizes $\mathbf{v}^T \mathbf{B}_u \mathbf{u}$ by opposing $\mathbf{B}_u^T \mathbf{v}$.

This yields the boundary flux functional:

$$\Psi(\mathbf{v}) = - \mathbf{v}^T \mathbf{D} \mathbf{v}$$

$$- u_{\max} * |\mathbf{B}_u^T \mathbf{v}|$$

$$+ w_{\max} * |\mathbf{B}_w^T \mathbf{v}|$$

Robust persistence requires:

$$\Psi(\mathbf{v}) \leq 0$$

for all $\mathbf{v}$ such that $E(\mathbf{q}, \mathbf{v}) = E_{\max}$.

A.3 Finite-Time Exit by Adversarial Disturbance

Assume there exists a boundary velocity $\mathbf{v}_{\star}$ such that:

$$\Psi(\mathbf{v}_{\star}) > 0.$$

By continuity of Ψ and compactness of the energy boundary, there exists $\epsilon > 0$ and a neighborhood U of $\mathbf{v}_{\star}$ such that:

$$\Psi(\mathbf{v}) \geq \epsilon$$

for all v in U .

Define the disturbance strategy:

$$w(t) = w_{\max} * \text{sign}(B_w^T v(t))$$

which is admissible and maximizes instantaneous energy injection.

Along any trajectory that enters U with positive measure in time, the energy derivative satisfies:

$$dE/dt \geq \epsilon$$

on a set of times of nonzero measure.

Integrating in time yields:

$$E(t) > E_{\max}$$

in finite time.

Therefore, no admissible control law can prevent exit from K once the inequality $\Psi(v) > 0$ holds at any boundary point.

This completes the proof of the negative law of robust persistence.